Banzhaf Power

Lecture 14
Section 2.2

Robb T. Koether

Hampden-Sydney College

Wed, Feb 14, 2018

Outline

Coalitions

Definition (Coalition)

A coalition is a group of players who agree to vote as a block. A winning coalition is a coalition whose votes add up to at least the quota. A losing coalition is a coalition whose votes add up to less than the quota.

Number of Coalitions

- If there are 3 players, how many possible coalitions are there?

Number of Coalitions

- If there are 3 players, how many possible coalitions are there?
- How about 4 players?

Number of Coalitions

- If there are 3 players, how many possible coalitions are there?
- How about 4 players?
- 5 players?

Number of Coalitions

xxx

(Nobody)
A
B
AB

2 Players, 4 coalitions

Number of Coalitions

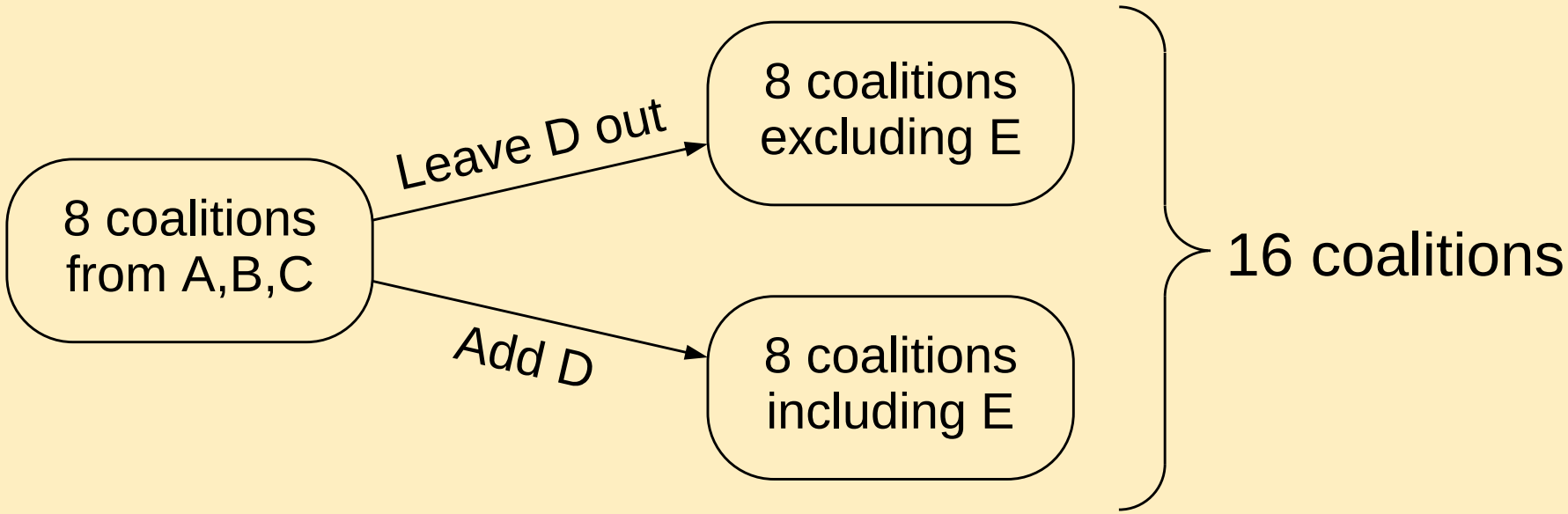
xxx

(Nobody)
A
B
C
AB
AC
BC
ABC

3 Players, 8 coalitions

Number of Coalitions

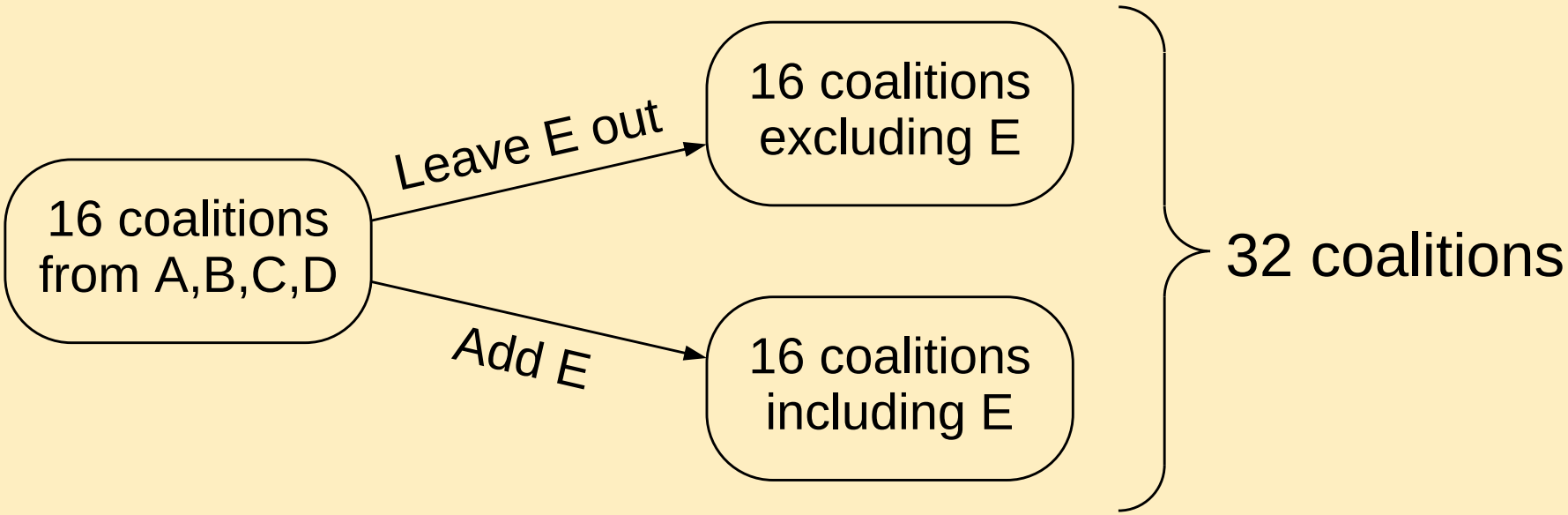
xxx



4 Players, 16 coalitions

Number of Coalitions

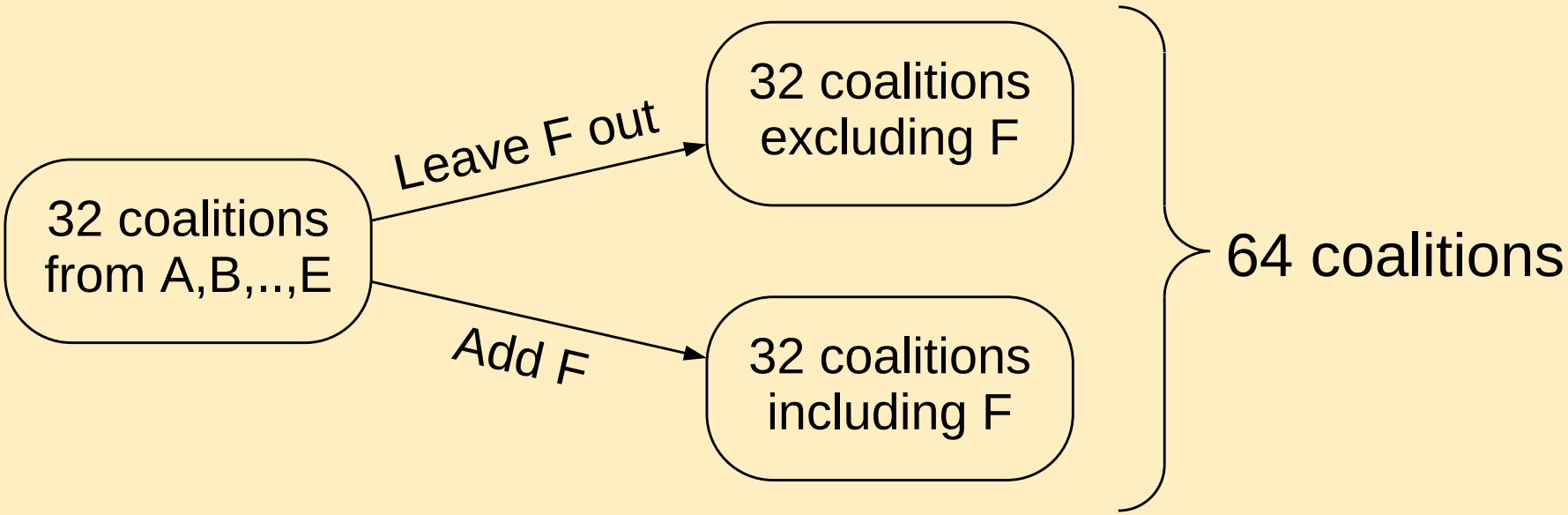
xxx



5 Players, 32 coalitions

Number of Coalitions

xxx



6 Players, 64 coalitions

Number of Coalitions

- Everytime we add one more player, the number of coalitions doubles (counting the empty coalition).
- Thus, if there are N players, then there are 2^N coalitions.
- What if there were 15 players?

Number of Coalitions

- Everytime we add one more player, the number of coalitions doubles (counting the empty coalition).
- Thus, if there are N players, then there are 2^N coalitions.
- What if there were 15 players?
- Then there would be $2^{15} = 32{,}768$ coalitions.

Number of Coalitions

- Everytime we add one more player, the number of coalitions doubles (counting the empty coalition).
- Thus, if there are N players, then there are 2^N coalitions.
- What if there were 15 players?
- Then there would be $2^{15} = 32{,}768$ coalitions.
- If there were 25 players, there would be $2^{25} = 33{,}554{,}432$ coalitions.

Listing Coalitions

- The best way to list the possible coalitions is by size.
  - Start with the empty set (or skip it).
  - Consider all coalitions of a single player: A, B, C, ...
  - Then consider all coalitions of two players by adding a player to the singleton coalitions: AB, AC, BC, ...
  - Then coalitions of three players, then four players, and so on.

Listing Coalitions By Size

By Size

$\varnothing$

The empty coalition

Listing Coalitions By Size

By Size

	A
∅	B
	C

Coalitions of size 1

Listing Coalitions By Size

By Size

	A	AB
$\varnothing$	B	AC
	C	BC

Coalitions of size 2

Listing Coalitions By Size

By Size

	A	AB	
∅	B	AC	ABC
	C	BC	

Coalition of size 3

Outline

Critical Players

Definition (Critical Player)

A critical player of a coalition is a player whose membership in that coalition takes it from a losing coalition to a winning coalition. That is, it is a winning coalition, but if he left it, then it would be a losing coalition.

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
A		
B		
C		
AB		
AC		
BC		
ABC		

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
A	4	
B	2	
C	1	
AB	6	
AC	5	
BC	3	
ABC	7	

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
A	4	
B	2	
C	1	
AB	6	A, B
AC	5	A, C
BC	3	
ABC	7	A

An Example

Example (Coalitions)

Consider the voting system $[5 : 4, 2, 1]$. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
A	4	
B	2	
C	1	
AB	6	A, B
AC	5	A, C
BC	3	
ABC	7	A

Notice that A has veto power, but A is not a dictator.

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
A		
B		
C		
AB		
AC		
BC		
ABC		

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
A	4	
B	2	
C	1	
AB	6	
AC	5	
BC	3	
ABC	7	

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
A	4	*A*
B	2	
C	1	
AB	6	*A*
AC	5	*A*
BC	3	
ABC	7	

Another Example

Example (Coalitions)

What if the quota were lowered to 4?

Coalition	Weight	Critical Players
A	4	*A*
B	2	
C	1	
AB	6	*A*
AC	5	*A*
BC	3	
ABC	7	

Now *A* is a dictator and *A* has veto power.

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
$\{A\}$		
$\{B\}$		
$\{C\}$		
$\{D\}$		
$\{A, B\}$		
$\{A, C\}$		
$\{A, D\}$		
$\{B, C\}$		
$\{B, D\}$		
$\{C, D\}$		
$\{A, B, C\}$		
$\{A, B, D\}$		
$\{A, C, D\}$		
$\{B, C, D\}$		
$\{A, B, C, D\}$		

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
$\{A\}$	9	
$\{B\}$	8	
$\{C\}$	3	
$\{D\}$	1	
$\{A, B\}$	17	
$\{A, C\}$	12	
$\{A, D\}$	10	
$\{B, C\}$	11	
$\{B, D\}$	9	
$\{C, D\}$	4	
$\{A, B, C\}$	20	
$\{A, B, D\}$	18	
$\{A, C, D\}$	13	
$\{B, C, D\}$	12	
$\{A, B, C, D\}$	21	

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
$\{A\}$	9	
$\{B\}$	8	
$\{C\}$	3	
$\{D\}$	1	
$\{A, B\}$	17	A, B
$\{A, C\}$	12	A, C
$\{A, D\}$	10	
$\{B, C\}$	11	B, C
$\{B, D\}$	9	
$\{C, D\}$	4	
$\{A, B, C\}$	20	(none)
$\{A, B, D\}$	18	AB
$\{A, C, D\}$	13	AC
$\{B, C, D\}$	12	BC
$\{A, B, C, D\}$	21	(none)

An Example

Example (Coalitions)

Consider the voting system $[11 : 9, 8, 3, 1]$.

Coalition	Weight	Critical Players
$\{A\}$	9	
$\{B\}$	8	
$\{C\}$	3	
$\{D\}$	1	
$\{A, B\}$	17	A, B
$\{A, C\}$	12	A, C
$\{A, D\}$	10	
$\{B, C\}$	11	B, C
$\{B, D\}$	9	
$\{C, D\}$	4	
$\{A, B, C\}$	20	(none)
$\{A, B, D\}$	18	AB
$\{A, C, D\}$	13	AC
$\{B, C, D\}$	12	BC
$\{A, B, C, D\}$	21	(none)

Notice that A has veto power, but A is not a dictator.

Outline

Definitions

Definition (Critical Count)

The critical count of a player is the number of possible coalitions in which he is a critical player.

Definition (Banzhaf Power Index)

The Banzhaf power index (BPI) of a player is that player's critical count divided by the total of all players' critical counts.

Definition (Banzhaf Power Distribution)

The Banzhaf power distribution is the set of BPI's for all the players.

Outline

Example

Example

- Find the power distribution in $[14 : 9, 8, 3, 1]$.

Example

Example

- Find the power distribution in $[14 : 9, 8, 3, 1]$.
- Does this sound right?

Example

Example

- Find the power distribution in $[11 : 9, 8, 3, 1]$.

Example

Example

- Find the power distribution in $[11 : 9, 8, 3, 1]$.
- Does this sound right?

Example

Example

- Use the Javascript program to find the Banzhaf Power Indexes in the following situations.
- $[14 : 6, 5, 5, 4]$.
- $[15 : 6, 5, 5, 4]$.
- $[16 : 6, 5, 5, 4]$.
- $[17 : 6, 5, 5, 4]$.

Example

Example (Stolen from Wikipedia)

- California has 55 electoral votes, Texas as 34, and New York as 31.
- Total = 120.
- If those were the only three states, then we would have $[61 : 55, 34, 31]$.
- Find the power distribution.

Example

Example (Also stolen from Wikipedia)

- Replace New York with Ohio, with 20 electoral votes.
- Total = 109.
- The situation now is $[55 : 55, 34, 20]$.
- How has the power distribution changed?

Example

Example

- Find the power distribution in $[9 : 5, 4, 3, 2, 1]$.
- You are E and you would like to buy one vote from another player. From which player should you buy it?

Example

Example

- Consider the situation $[q : 3, 3, 2, 1]$.
- What quota(s) q makes the power distribution most balanced?
- What quota(s) q makes the power distribution most unbalanced?

Outline

Assignment

Assignment

- Chapter 2: Exercises 11, 12, 13, 14, 15, 17, 19; 69, 71. (You may want to use the Javascript program for 69 and 71.)